ETEAM

Problems for the second ETEAM

VERSION 1.3 UPDATED ON APRIL 29, 2025

Foreword

The problems that follow are difficult and are proposed by researchers and students in mathematics. To the best of the knowledge of the authors, they do not always admit a complete solution. However, they are accessible to high school students, i.e., the authors are certain that elementary research work can be carried out on these problems. The jury does not expect the candidates to solve a problem entirely, but rather to understand the issues, solve particular cases, identify difficulties, and suggest directions of research. The questions are not always arranged in increasing order of difficulty. Finally, it is not necessary to solve all the problems: each team can reject a certain number of problems without penalty. Please refer to the rules for further details.

Contents

Keywords

1. Game theory 2. Combinatorics, Probability 3. Probability 4. Optimization 5. Graphs 6. Geometry, Combinatorics 7. Number theory 8. Probability, Dynamical systems 9. Combinatorics, Dynamical systems 10. Word sequences

Notations

$\mathbb{R}, \mathbb{Z}$	The sets of real numbers and integer numbers, respectively
$[a,b)$	The set of all numbers $x \in \mathbb{R}$ such that $a \leqslant x < b$
$\mathbb{N}$	The set of strictly positive integer numbers $\{1,2,\dots\}$
$[\![n,m]\!]$	The set of integers $\{n,n+1,\dots,m\}$

1. Game on flowers

To keep busy during their long travel to come to Lyon for the ETEAM, Aster and Lily decide to play a game. They both like the *Mill*, but find the usual board a bit boring, so they decide to play on a fancier shape. They alternatively colour one vertex of a flower-shaped graph, with the aim to form a certain one-coloured **winning configuration**, that will be defined later. Aster starts, and when a player forms the winning configuration, the game stops.

Let $n \geq 3$ and $k \geqslant 1$ be two integers. A **flower graph** of size n with k layers consists of $k+1$ concentric circles, on each of which there are n marked points. Finally, between neighbouring circles, the points are linked in a triangular way, each point of a layer being linked to two points in every neighbouring layer; see Figure 1, where the red player has so far achieved two triangles, and the blue player a path (and even a cycle, going around the inner hole) of length 6. A player has a **winning strategy** when by playing the right moves he can win no matter what the other player is doing.

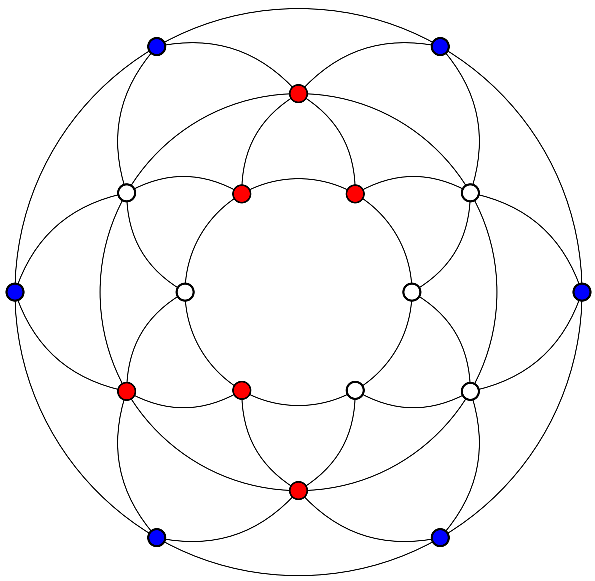

FIGURE 1. A flower graph of size 6 with 2 layers

For the first questions, we consider only $k = 1$.

1. The winning configuration is a triangle. Does any player have a winning strategy?

2. The winning configuration is now either a triangle or a circuit of length n, i.e., a player wins whenever they achieve at least one of those two configurations. Does any player have a winning strategy?

Now, we consider also flower graphs with more layers.

3. What is the longest path Aster can ensure to make on a general flower graph of size n with k layers? What about Lily? One could start with $k = 1$ and $k = 2$.

4. Describe all winning configurations for which Lily has a winning strategy.

Lily changes the rules slightly: instead of stopping the game when a player achieves the winning configuration, the game only stops when all vertices have been coloured. If exactly one player achieved the winning strategy, they win otherwise it is a draw.

5. Reconsider the previous questions with the new rule proposed by Lily.

Aster proposes another change of the rules: instead of colouring vertices, the players now colour the edges of the graph. The game stops again when one player achieves the winning configuration. For instance, on Figure 2, the blue player has so far achieved one cycle going around the inner hole, while the red player has formed two bow ties.

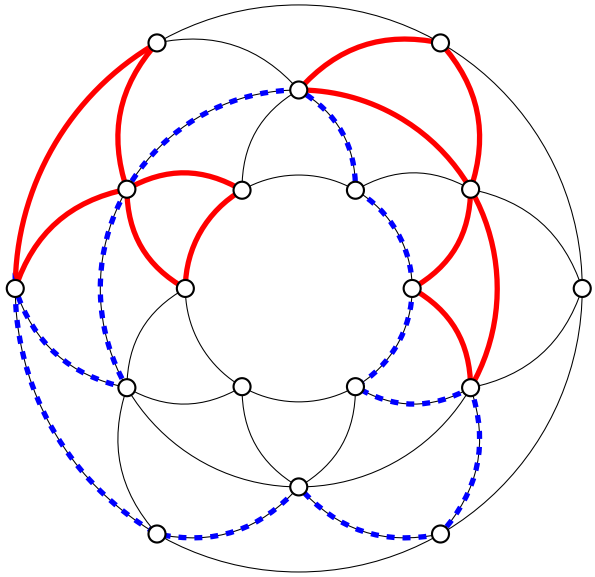

Figure 2. An edge-colouring game on the flower graph of size 6 with 2 layers

6. The winning configuration is a cycle going around the inner hole. Does any player have a winning strategy? One can start with $k = 1$.

7. The winning configuration is a bow tie, this is two triangles with one common vertex ($\bowtie$). Does any player have a winning strategy?

Lily's change applies again: both players continue until all edges have been coloured.

8. Reconsider the two previous questions under Lily's variant of the rules. What happens if at the end of the game, each player counts how often they achieved the winning configuration and the winner is the person who did the most?

9. Suggest and study other research directions.

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2. Blanket falling

We consider a flat piece of foil and we would like to make a modelling of the fall of this foil onto the floor. We first fix the area of the foil throughout this exercise no matter what the shape is. We also assume that it is infinitely thin. The folding of the foil is modelled by the folding process of the following form. A **process** P is a finite sequence of pairs (F_k, L_k) where F_k is a (folded) flat piece of foil and L_k is either the empty set or a line on the foil. If L_k is not the empty set then we obtain F_{k+1} from F_k by folding F_k along the line L_k. The finite sequence must end with a pair $(F_k, \emptyset)$, and in this case, F_k is the final shape of the folding process.

1. In this exercise, we assume that the foil is of a rectangular shape and can only be folded along the lines of a **rectangular grid** $n \times m$ divided in nm squares as in Figure 3. Folding along such a line, one obtains again a rectangular piece of foil with an $n' \times m'$ grid of lines along which we can fold again.

a) What are the possible shapes the final form can be?
b) How many different folding processes are there?
c) What is the shape that can be obtained by the most number of folding processes? In how many ways can it be obtained?

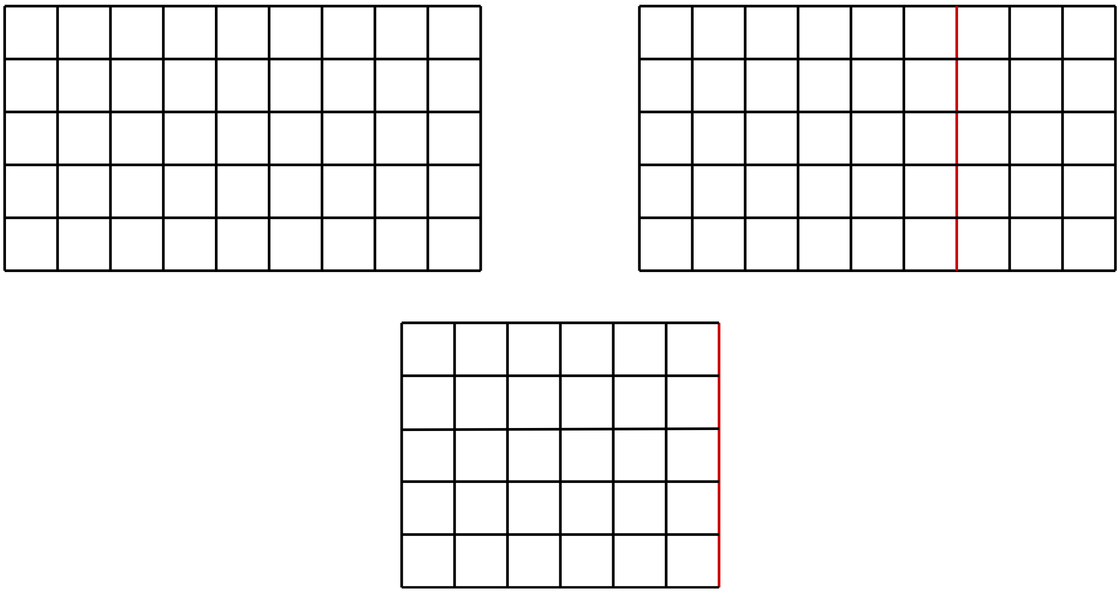

FIGURE 3. On the top left there is a 5×9 grid (F_0), on the top right we take L_0 as the red line and at the bottom there is resulting foil (F_1) after we folded along the line L_0, the resulting foil is a 5×6 grid

d) We assume that every folding process is equally probable to happen. What is the expected area of the final shape?

2. We now consider a foil of the shape that is an **equilateral triangle** where we can fold along lines that form a triangular pattern and n lines meeting each side (see the left of Figure 4). Answer the questions of **1.** in this case.

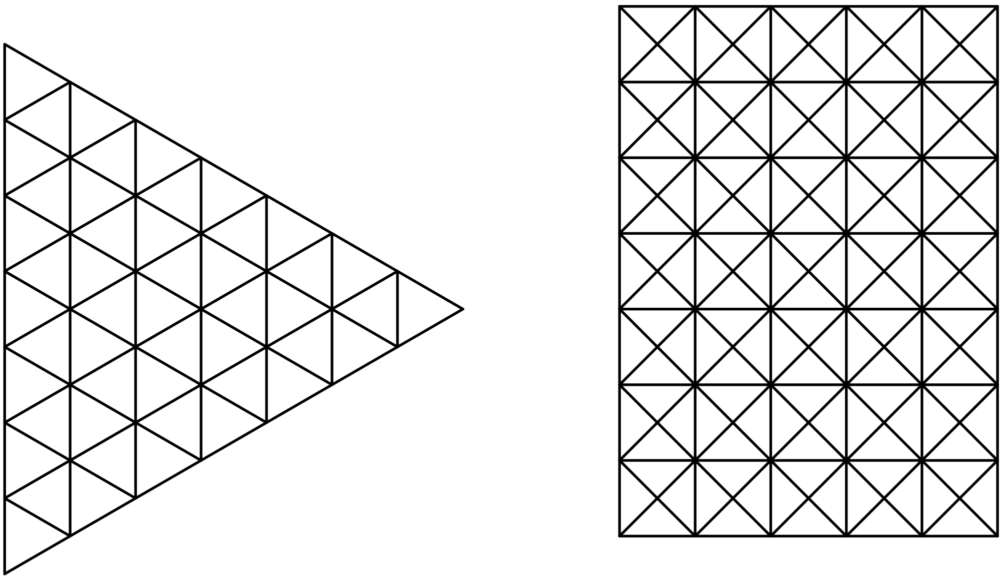

FIGURE 4. A triangular foil of size 7 and a rectangular foil of size 7×5 where we are also allowed to fold along the diagonals

3. We now assume that the shape of the foil is a square with an $n \times m$ grid, but where it can also fold along diagonal lines as on the right of Figure 4. Answer the question in **1.** in this case.

4. We are again in the situation of question **2.**. For each n, m and $b \in \mathbb{N}$ what is the minimal length of a folding process such that the resulting shape has b boundary components?

5. Assume that we start folding a piece of foil that has area 1. For a folding process $P = \{(F_k, L_k)\}_{k=1,\dots,N}$ we let $A(P)$ denote the **area** and $B(P)$ denote the **length of the boundary** of the resulting shape. In the situation of Questions **1.**, **2.** and **3.** try to find the maximal value and the expected value (we assume here that each folding process has the same probability of occurring) of the quantity

$$\frac{A(P)}{B(P)^2}$$

and find the shape that maximizes this quantity. How does this quantity behave as the grid gets finer and finer divisions?

6. Suggest and study other research directions.

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3. Dropping intervals

We would like to study the following situation. Let S^1 be a circle of circumference 1 on which n intervals of length $0 \leqslant \alpha \leqslant 1$ are dropped with uniform probability. This situation looks like too complicated at first so we will start with a 'discretized' version of this problem.

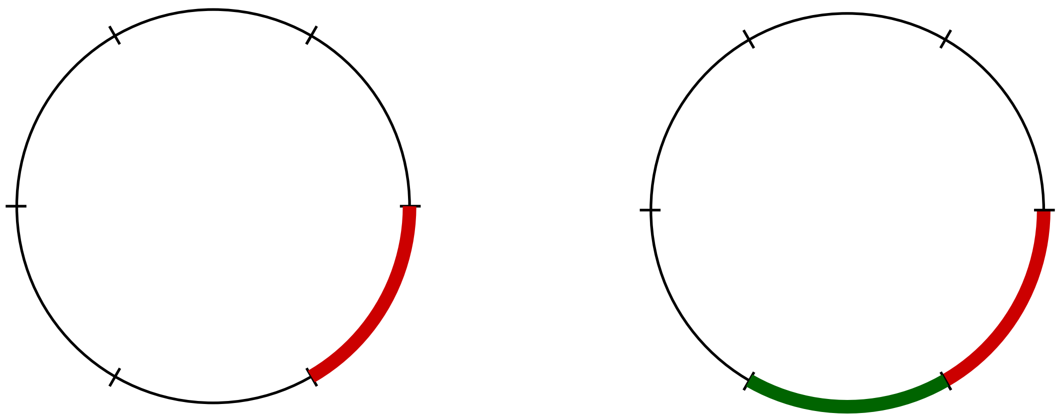

FIGURE 5. A picture of a circle split in 6 intervals. On the left a first interval is dropped in red and on the picture on the right a second one in green. On the right, the red and green interval form a block

1. We split the circle S^1 in k intervals I_1, I_2, ..., I_k of equal length $1/k$, called **cells**. We drop n intervals of length $1/k$ on S^1. First, we assume that each of the dropped intervals covers exactly one cell I_i, where i is chosen following a uniform law on $[\![1, k]\!]$, see Figure 5. We want to know the expectancy of the following variables, depending on k and n.

a) What is the expectancy of the total length covered by the dropped intervals?
b) Fix a sequence of ℓ adjacent cells in S^1. What is the probability that these cells are completely covered by the dropped intervals?
c) We say that a maximal collection B of cells all adjacent is a **block**. What is the expectancy of the number of blocks after we dropped the n intervals?

2. In this question the only difference is that each interval dropped has length m/k for some fixed integer $m < k$, and covers exactly m cells. Answer questions **1.** a), b), and c) in this new setting. The answers should be written in terms of the values of k, n, and m.

3. We stay with the hypothesis of the previous question, what can you say about the answers of the previous question when the integers k and m increase to infinity while the rational number $\alpha = m/k$ remains constant? In other words we take smaller and smaller subdivisions of the circle S^1 but we fix the length α of the intervals that drop.

4. We have c colours that we name C_0, C_1,..., C_{c-1}. We throw again n intervals of length m/k inside the circle S^1 split as before in cells I_1, I_2, ..., I_k of equal length $1/k$. At the beginning all the cells have the colour C_0. Each time an interval of length m/k falls on some set of cells of colour C_a, each of the m touched cells changes colour and becomes of colour C_{a+1} (if $a = c-1$, then we let $C_c = C_0$).

a) Compute the expectancy of the total length of each colour in terms of n, m, c and k. One could start by answering the question in the setting of question **1.**.
b) Study the way the random variable $X_j(n)$ (that is the length coloured by C_j after n throws) behaves as n goes to $+\infty$.

5. Up to now we have 'discretized' the circle S^1. We want to know what happens when we drop intervals of fixed length $\alpha \leqslant 1$ centred on a point chosen anywhere in the circle S^1 with uniform probability. Discuss the previous questions in this new setting.

6. Suggest and study other research directions.

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4. The apprentice and the witch

A witch is forging spells by assembling runes on her workbench. Her workbench looks like a huge square plate that we identify with $[1;\ell]^2$ for some integer $\ell \geqslant 1$, with a notch for a rune on each point with integer coordinates. She starts by preparing her workbench by inserting runes in some of the notches. Each notch may contain no or one rune. The description of the way the notches are filled is called the **initial configuration**. Moreover, each rune has a **weight**, which depends on the type of the rune.

Once the initial configuration is set, the witch may no longer add or remove runes. She may only move them in order to combine them, respecting the following rule: she moves runes one by one, in a horizontal or vertical line, only from one notch to another. (One rune may be moved several times, following a path formed by a broken line. A rune cannot jump above another.) The **mana cost** of such an operation is the product of the distance being travelled by the weight of the rune.

1. The witch starts by forging healing spells. She works only with one type of rune, of weight equal to 1. Once a rune is moved into a notch containing another rune, they merge to create one healing spell, and vanish from the board. For instance, after the move in Figure 6, the notches A and B will be empty, and the witch will have one healing spell in hand. In the initial configuration, every notch of the workbench is filled by a rune.

a) How many healing spells can she create from this initial configuration?
b) What is the minimum mana cost needed to create those spells?
c) The witch wants to play a bad trick to one of her enemies, who asked her the recipe for the healing spell, by indicating her the most expensive way to create healing spells. To make the trick not so obvious to detect, the same configuration should not arise twice on the board. What is the maximum mana cost that can be achieved this way? Try to give an estimation as precise as possible.

2. During the night, the child apprentice of the witch comes in her workshop, and to prank her, he picks up some runes and hides them in the garden. The next day, the witch thus has to start with a new initial configuration, where some notches have been emptied by the apprentice. In the following situations, how many healing spells may she create each day, and what is the

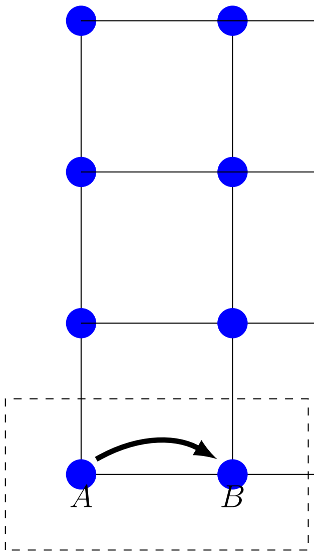

FIGURE 6. A workbench with each notch filled, for $\ell = 4$. The rune in A moves to the notch B, travelling a distance of 1

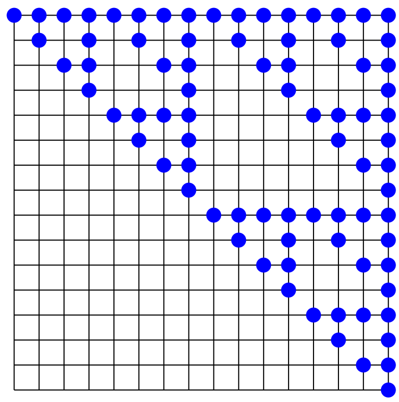

FIGURE 7. The workbench at the end of the second night (Question 2b).

minimum mana cost for doing so? Of particular interest is the asymptotic ratio between the number of spells and the mana cost when $\ell \to +\infty$, in case no explicit formula is available.

a) The first night, the apprentice takes out every rune which is in a notch whose coordinates are either both even or both odd.
b) The second night, the apprentice is faced with a new fully filled workbench whose side length ℓ is a power of two, and finds himself to be especially inspired. He starts by dividing the whole workbench into four equal squares, and picks up all the runes that are in the bottom left one. Then, he divides the three remaining squares into four equal smaller squares, and in each of them, he picks up all runes in the bottom left one. He continues so until he finds himself dividing squares of side length 2 into squares of side length 1, at which moments he picks the bottom left rune in each square and leaves. See Figure 7 for an illustration with $\ell = 16$.

3. We denote by n the number of runes that are left on the bench after the visit of the apprentice. Given a configuration, we denote by C the minimal mana cost needed to merge all runes.

a) What are the possible values of n/C?
b) If $n = 2k$, how can the apprentice choose how to remove the runes in order to maximize the mana cost of creating k healing spells? (You can start by studying $k = 1,2,3$ and $\ell = 2k$.)

4. The witch now wants to forge a powerful blasting spell. She works with new runes, which are labelled by a positive integer number, called their **power**. In the initial configuration, every rune has a power of 1. When two runes come together, they merge into a new rune whose power is the sum of the powers of both runes. The weight of a rune of power k is k^α, with $\alpha \geqslant 0$ given. For instance, after the move in Figure 6, the notch A will be empty, and the notch B will contain a rune of power 2 and weight 2^α. The goal of the witch is to obtain a rune with the highest possible power, to get a devastating blasting spell. First, assume that the witch starts from the initial configuration where every notch contains a rune.

a) What is the minimal mana cost of forging a rune of power ℓ^2? You can start by studying the cases $\alpha = 0$, $\alpha = 1$. In case it is impossible to obtain an exact formula, it will be of particular interest to study the asymptotic of the ratio of the mana cost by the power of the rune that is obtained as $\ell \to +\infty$.

b) Fix ℓ and let $\alpha \to +\infty$. What is the behaviour of the total mana cost of forging a rune of power ℓ^2? Try to give an asymptote as precise as possible.

c) Now, explore the general case where the initial configuration is not necessarily full. You may consider the examples of Question 2.

5. In the previous situation, the weight of a rune of power k is given more generally by $f(k)$, for some function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ (The previous situation corresponds to $f(x) = x^\alpha$.) Try to generalise the answers to the previous question for various shapes of the function f, as general as possible.

6. In order to create a fire spell, the witch works with slightly different runes than in the previous situation. This time, each rune carries not a number, but a set in $\mathbb{R}^2$. In the initial configuration, every rune carries the square of side length 1 whose centre is the notch the rune starts in. When two runes merge, they form a new rune which carries the union of the sets carried by both runes. The **power** of a rune is the area of the set it carries, while its weight is the perimeter of this set. For instance, after the move in Figure 6, the notch B will contain a rune whose associated set is the 2×1 dashed rectangle, hence with power 2 and weight 6. What is the minimal mana cost of forging a rune of power ℓ^2? In case it is impossible to obtain an exact formula, it will be of particular interest to study the asymptote of the ratio of the mana cost by the power of the rune that is obtained as $\ell \to +\infty$. One may start from the initial configuration where every notch contains a rune, and then try to explore the general case.

7. Suggest and study other research directions.

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5. Graphs and special colourings

In this problem, by a graph we always mean an undirected graph $G = (V, E)$, where V is a set of vertices and $E \subseteq V \times V$ is a set of edges together with a fixed numbering of the set V, i.e., a bijection $d : \{1, 2, \ldots, |V|\} \to V$. An **isomorphism of graphs** $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a bijection $\varphi : V_1 \to V_2$ such that $(v_1, v_2) \in E_1$ if and only if $(\varphi(v_1), \varphi(v_2)) \in E_2$. An isomorphism of graphs φ is called **strong** if, in addition, it satisfies $\varphi \circ d_1 = d_2$.

For a given positive integer c, a **c-colouring** of a graph G is a function $f : V \to \{1, \ldots, c\}$. We call the pair (G, f) a **c-coloured graph**. We consider two c-coloured graphs (G_1, f_1) and (G_2, f_2) to be **isomorphic** (resp. **strongly isomorphic**) if there exists an isomorphism (resp. a strong isomorphism) of graphs $\varphi : G_1 \to G_2$ and a c-permutation $\pi \in S_c$ such that for any vertex $v \in V_1$ we have $f_1(v) = \pi(f_2(\varphi(v)))$. For example, the two coloured graphs below are isomorphic but not strongly isomorphic.

In this problem, we are interested in enumerating coloured paths and cycles, possibly up to isomorphism and strong isomorphism. A **path graph** on n vertices is a connected graph with vertices $V = \{v_1, \ldots, v_n\}$ and edges $E = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)\}$. For instance, a path graph on 6 vertices is given below in Figure 9.

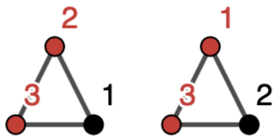

FIGURE 8. On the left, the vertices 2 and 3 are coloured in red while the vertex 1 is coloured in black. On the right, it is the vertex 2 that is coloured in black while the others are coloured in red.

FIGURE 9. The path graph on 6 vertices.

1. By P_n we denote the path graph on n vertices with any fixed numbering of its vertices d_n.

a) Find the number of 2-coloured graphs (P_9, f) up to (strong) isomorphism (i.e., one part of the question concerns counting up to isomorphism and another up to strong isomorphism).

b) Find the number of 2-coloured graphs (P_n, f) up to (strong) isomorphism as a function of n.

c) Find the number of 3-coloured graphs (P_9, f) up to (strong) isomorphism.

d) Find the number of 3-coloured graphs (P_n, f) up to (strong) isomorphism as a function of n.

e) Find the number of c-coloured graphs (P_n, f) up to (strong) isomorphism as a function of c, n.

2. An m**-chain** in the c-coloured graph (G, f) is a path subgraph $(V_H, E_H) =: H \subseteq G$ with m vertices such that there exists $1 \leqslant i \leqslant c$ with $f(v) = i$ for any $v \in V_H$.

a) Find the number of 4-coloured graphs (P_9, f) without 3-chains.

b) Find the number of 4-coloured graphs (P_n, f) without 3-chains as a function of n.

c) Find the number of c-coloured graphs (P_n, f) without m-chains as a function of m, c, n.

d) Study the above questions considering graphs up to (strong) isomorphism.

3. An m**-antichain** in the c-coloured graph (G, f) is a path subgraph $(V_H, E_H) =: H \subseteq G$ with m vertices such that for any two connected vertices $v_1, v_2 \in V_H$ we have $f(v_1) \neq f(v_2)$.

a) Find the number of 2-coloured graphs (P_n, f) without m-antichains as a function of n, m.

b) Find the number of c-coloured graphs (P_n, f) without m-antichains as a function of n, m, c.

c) Find the number of 2-coloured graphs (P_n, f) without m-antichains and without m-chains as a function of n, m.

d) Find the number of c-coloured graphs (P_n, f) without m-antichains and without m-chains as a function of n, m, c.

e) Study the above questions considering graphs up to (strong) isomorphism.

4. A **strong** m**-antichain** in the c-coloured graph (G, f) is a path sub-graph $(V_H, E_H) =: H \subseteq G$ with m vertices such that for any two vertices $v_1, v_2 \in V_H$ we have $f(v_1) \neq f(v_2)$. Study question **3** for strong antichains.

5. By C_n we denote the cycle on n vertices. Study questions **1-4** for C_n instead of P_n.

6. Study questions **1-5** for other classes of graphs.

7. Suggest and study other research directions.

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6. Integer polygons

Let N be a positive integer. We consider the set T_N of triangles with natural integer side lengths a, b, c which are at most N, and up to rotations and reflections. This means that two triangles are considered the same when they are isometric.

1. How many elements are in T_5? Among them how many are
a) isosceles?
b) right triangle?
c) acute?

2. How many elements are in T_N for general N?

3. What is the proportion of acute triangles within elements of T_N? How does this quantity behave when N goes to infinity?

A puzzle fan would like to use all elements of T_N to build a 2-dimensional non-overlapping connected shape.

4. a) Is that possible?
b) Is that possible if we add the constraint that two triangles can only be neighbours when they share a common edge (called "**edge-to-edge**")?

Some triangles in T_N are similar to each other. We call T'_N the set of natural integer length triangles with side lengths at most N, up to rotations, reflections, and homotheties.

5. Reconsider the previous questions for T'_N.

Instead of triangles, we now consider convex n-gons which are inscribable in a circle. Let $n \geqslant 4$ be a fixed integer. Let $B_N^{(n)}$ denote the set of $(a_1, a_2, ..., a_n) \in [\![1, N]\!]^n$ such that there is an inscribable polygon with side lengths $a_1, ..., a_n$ (in that order) up to rotations and reflections.

6. How many elements are in $B_N^{(4)}$? How many elements are in $B_N^{(n)}$ in general?

7. What is the proportion of n-gons in $B_N^{(n)}$ for which the centre of the circumcircle lies strictly inside the n-gon? How does this quantity behave when N goes to infinity while n is fixed? How does it behave when n goes to infinity while N is fixed?

8. Is it possible to lay down all elements of $B_N^{(n)}$ without overlaps and edge-to-edge to form a connected shape of the plane?

9. Suggest and study other research directions.

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7. Strange lifts

Imagine a building with infinitely many storeys. The storeys in this building are numbered by the non-negative integers in increasing order. Alice is standing on the zero-th storey of the building and she wants to meet with Bob, who lives on the N-th storey for some $N \in \mathbb{N}$.

Alice wants to use the building's lift, however there is one rule for using it. There is a set of numbers $A \subset \mathbb{N}$ with the only condition that $1 \in A$. Alice is allowed to go on the lift only from the k-th storey to the $(k+a)$-th storey for any $a \in A$.

1. Alice wants to estimate how long she will travel from storey to storey to get to Bob's apartment. She denotes the number of lift uses required to get to Bob's apartment as $d_A(N)$.

a) Can Alice always reach Bob's apartment? When it is possible to reach Bob's apartment, give a bound on $d_A(N)$ independent of A.

b) Decide whether $d_A(N+M)$ is always less than or equal to $d_A(N)+d_A(M)$ or not.

2. Alice understands that deriving a formula for $d_A(N)$ can be hard in many cases. Thus, in general she is interested in approximations for d_A as a function of N. She says that two functions $f,g:\mathbb{N}\to\mathbb{N}$ are **approximately equal** (or that f is approximately g) and writes it as $f\approx g$ if there is a constant $C>0$ such that for all $n\in\mathbb{N}$,

$$\frac{1}{C}g(n)-C\leqslant f(n)\leqslant Cg(n)+C.$$

Try to calculate d_A precisely or give an approximation in the following cases:

a) $A=\{1,a,b\}$ for $a,b\in\mathbb{N}$ arbitrary.
b) A is an arbitrary finite set.
c) $A=\{1\}\cup\{P(n)\}_{n\in\mathbb{Z}_{\geqslant 0}}$ where $P\in\mathbb{Z}[x]$ is a polynomial with non-negative integer coefficients.

3. Alice wants to know for which sets A the value of $d_A(N)$ is bounded from above for all N.

a) Does there exist a function $h:\mathbb{N}\to\mathbb{N}$ such that if $A=\{1\}\cup\{x_1,x_2,\ldots\}$ with $x_i\geqslant h(i)$ for all $i\in\mathbb{N}$, then $d_A\not\approx 0$?
b) Does there exist $p>1$ such that the function d_A for $A=\{1\}\cup\{\lfloor p^n\rfloor\mid n\geqslant 1\}$ is not approximately 0?
c) Try to characterize all sets A for which $d_A(N)$ is bounded from above for all N.

4. For a real number $t\geqslant 1$ denote $A_t=\{1\}\cup\{\lfloor t^n\rfloor\mid n\geqslant 1\}$. Alice finds sets A_t intriguing so she wants to understand more about different speeds of approaching Bob for them. Try to calculate d_A precisely or give an approximation in the following cases:

a) $A=A_2$ and $A=A_3$.
b) $A=A_t$ for $t\in\mathbb{N}$.
c) $A=A_t$ for any $t\in\mathbb{R}_{\geqslant 1}$.

5. Can you find an infinite set $T\subseteq\mathbb{R}_{\geqslant 1}$ such that for any $t,t'\in T$ with $t\neq t'$ we have $d_{A_t}\not\approx d_{A_{t'}}$? Can you find such an uncountable set T?

6. Can you calculate or approximate d_A for some other sets A? For example, for the set of Fibonacci numbers $A=\{1,2,3,5,8,13,\ldots\}$.

7. Suggest and study other research directions.

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8. WANDERING BACTERIA

The bacteria *E.tim* are a kind of bacteria that live in a very dynamic pluricellular mode. The bacteria live stuck together in a 1D structure as a "stick" (**Figure 10 and 11**). This colony can grow and shrink over time. This evolution only happens at the ends of that colony, that is, over time a colony can grow or shrink by its ends with some probability rate. We denote by $0<\lambda\leqslant 1$ the probability of growth at each end of the colony. We denote by $0\leqslant\mu\leqslant 1$ the probability of shrinking of the colony at those said ends. In our model, we consider a discrete evolution in time. More precisely, this means that, at each time $n\in\mathbb{N}$, and at each end of the colony, there is a probability λ that the length increases by 1, and a probability μ that it decreases by 1. (If $\mu\neq 0$, it may happen that the colony grows and shrinks simultaneously, either both at the same end, or each at a different end, in which case it keeps the same length. Also, the colony may grow, or shrink, at both its ends at the same time, in which case the length would increase or decrease by 2.) We denote by L_n the length at time $n\in\mathbb{N}$, and we always assume that $L_0>0$. If for some n, $L_n=0$, the colony dies and the process stops.

1. We first wonder about the length of the colony as time elapses.

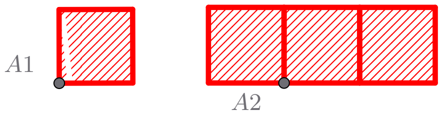

FIGURE 10. The colony grew by both sides between step 1 and step 2.

FIGURE 11. b) Case 2:The colony grew only by the right side

a) Assume that the colony can only grow with probability λ, that is, $\mu = 0$. How does the length of the colony evolve over time, depending on L_0? What is the expected length at time n?
b) Assume that $\mu \neq 0$. What is the expected length of the colony at time n, depending on L_0?

2. There is a toxin in the environment that kills bacteria. At each time, there is a probability κ that the colony enters in contact with the toxin, in which case, three consecutive bacteria die.

a) We first assume that the toxin only kills the three bacteria at the top end of the colony. Estimate as precisely as possible the probability that a colony dies.
b) From now on, we assume that the three consecutive bacteria killed by the toxin are taken uniformly at random in the colony. If they are not at the end of the colony, then the latter is split into two colonies, that evolve independently until they possibly reunite by coming again in contact with each other. Estimate as precisely as possible the probability that a colony dies.
c) Estimate as precisely as possible the length of the different colonies depending on time.

From now on, the colony evolves in a one-dimensional embedded in a 2 dimensional environment and can change its direction of growth (going to its right or to its left) by its end with probability ϵ. That is, whenever the colony grows by some of its ends, there is a probability ϵ that it will grow either on the right or on the left (chosen uniformly at random), and this direction becomes the new direction of growth until a new change happens. That is to say that for each step of the process, there are three possible directions for the bacteria population to grow.

3. We assume that, at time $n = 0$, the colony starts with only one bacteria, located at the point of coordinates $(0,0)$.

There is a food source at the point of coordinates (x,y). In each of the following situations, we wonder whether the colony will ever get to the food, and what would be the expected time at which it will happen.

a) We first assume that the colony may form (non lethal) loops, and we therefore assume that multiple bacteria can live at the same place, i.e, a bacteria can grow in a location where there was already another bacteria. In this case, the direction of growth of the population is the direction from the new bacteria.
b) We now assume that bacteria cannot overlap. That is, if a bacteria were to appear at a point where there is already another bacteria, then it dies immediately.
c) Finally, we assume that the food attracts the colony, that is, the colony has a trend to grow in the direction which minimises its distance to the food. This behaviour is modelled as follows. Among the three admissible directions (forward, left, right), the probability to go in one of them which would minimize the distance (remark that two directions can

minimize the distance) to the food among the others is given by $\frac{1}{3}+\frac{1}{d+3}$, where d is the current distance from the end of the colony to the food. Here, the distance between two points is defined as the minimum number of steps to go from one point to each other, namely, the distance between (x_1,y_1) and (x_2,y_2) is given by $|x_1-x_2|+|y_1-y_2|$. When two directions minimize the distance, the probability defined for each growing directions still is $\frac{1}{3}+\frac{1}{d+3}$. The probability to go in any direction which do not minimise the distance is then defined accordingly so that the probabilities sum to 1, and so that the probabilities to go in all these directions are equal.

In fact, researchers discovered that *E.tim* can evolve in a 2-dimensional model. A bacteria occupies a square of length 1. Those bacteria can form a colony while assembling by their sides. A bacteria can only grow next to another bacteria. At any time $n \in \mathbb{N}$, the probability that a new bacteria grows at an admissible empty place is given by $\lambda + (1-\lambda)\frac{k-1}{k}$, where k denotes the number of neighbouring places where there is already a bacteria. At time $n=0$, there is only one bacteria.

4. How does the amount of bacteria evolve with respect to time? What is the expected number of bacteria at time n?

5. In reality, there are two different species of bacteria that can grow: *E.tim2024* and *E.tim2025*, with respective probabilities of growth λ_1 and λ_2.

a) Those bacteria are competing. At time $n=0$, there is a *E.tim2024* in $(0,0)$ and a *E.tim2025* in $(10,10)$. If a bacteria *E.tim2024* is in contact with a *E.tim*2025, the bacteria *E.tim2024* dies. Will a population go extinct ?

b) We now assume that *E.tim2025* and *E.tim2024* do not kill each other but block entirely the growth of the other population, that is to say that a *E.tim2025* and *E.tim2024* can't overlap. When those bacteria are adjacent, they change their direction of growth uniformly in admissible directions (that is to say, directions that won't lead those two populations to overlap at the next step of the process). How will those populations behave asymptotically in time? You can try different initial condition for the distribution of populations at time $n=0$

6. Suggest and study other research directions.

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9. A game with light

Guillaume is playing with a special cube where each of its six faces contains an $n \times n$ grid of lights. Each light can be in one of three states: **red**, **blue**, or **green**. When Guillaume presses a light, its state changes cyclically as follows:

$$\text{Red} \to \text{Blue} \to \text{Green} \to \text{Red}.$$

However, pressing a light does not only affect itself! It also changes the state of its adjacent lights on the same face (not diagonals), as well as the lights directly connected across the edges of the cube. See Figure 12 for illustration.

Guillaume's goal is to turn all the lights **green**, but he quickly realizes that this is not an easy task! He wonders if there is a strategy to solve the puzzle efficiently.

1. Guillaume wonders whether the **order** in which he presses the lights affects the final result. If he presses the same set of lights but in a different sequence, will he always obtain the same final state?

2. In this question, Guillaume assumes that $n=2$. If all lights are red at the beginning, is it possible to turn all lights green?

3. In this question, Guillaume assumes that $n=2$. No matter the original state of the cube, is it possible to turn all lights green?

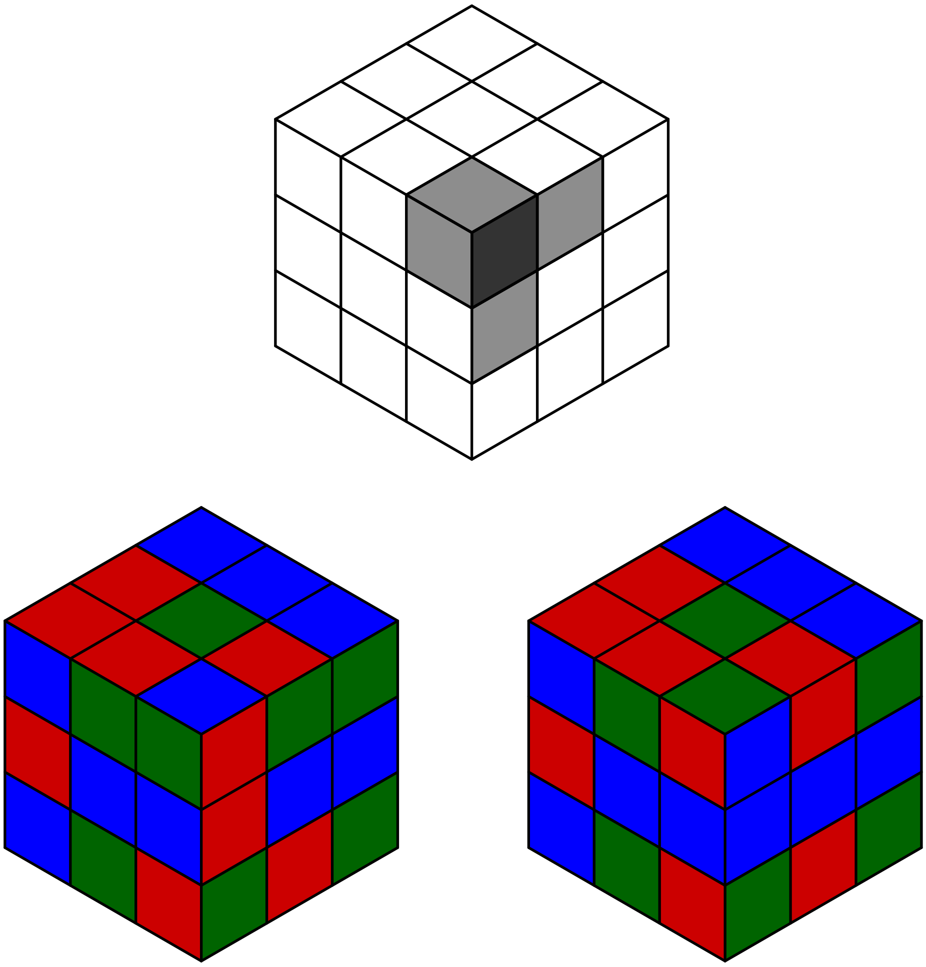

FIGURE 12. At the top, on the cube 3×3 if we press the square in dark grey we would change its colour as well as the one of all the square in light grey. Below you can see an example, on the left is the 'before' picture and on the right is the square after we press on the dark grey squares

4. What is the minimum number of moves that Guillaume needs to solve the worst-case scenario which has a solution on a 2×2 cube face?

5. Reconsider the previous question for arbitrary n?

6. How many unsolvable initial configurations are there as a function of n?

7. Now, instead of manually pressing the lights, the system evolves automatically according to a cellular automaton rule: a light remains unchanged unless it is influenced by at least four adjacent lights of the same colour.

- If a light is surrounded by at least four adjacent lights of the same colour, it advances in the cycle

$$\text{Red} \to \text{Blue} \to \text{Green} \to \text{Red}.$$

- If a light has exactly one red and one blue neighbour, it flips to the opposite state in the cycle.

a) For which initial conditions does the system evolve to a stationary configuration?
b) What is the longest time for the system to evolve to a periodic configuration?

8. Suggest and study other research directions.

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10. Play on words

Misha has the set $\{0,1\}^*$ of all finite words over the alphabet $\{0,1\}$, meaning the set $\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, ...\}$ with ε being the empty word. Misha notices the following, sometimes if you have a finite subset $S \subset \{0,1\}^*$, you can represent some elements of $\{0,1\}^*$ as concatenation of elements of S. We say that the set of all concatenations of words of S is **generated** by S and we denote it by S^*. For example if $S = \{00, 011\}$ we can concatenate the words of S to obtain the word $01100011 \in S^*$.

When $S = \{0,1\}$ we can represent all elements of $\{0,1\}^*$. Since it is easy to come up with these sets, Misha wants to understand a characteristic of this set. Define the **effectiveness** of S as:

$$\mathrm{Eff}(S) = \lim_{n\to\infty} |S^{\leqslant n}|^{\frac{1}{n}}$$

Where given a finite set A, $|A|$ is its cardinal. $S^{\leqslant n}$ denotes the set of words that are concatenation of at most n elements of S, that is to say:

$$S^{\leqslant n} = \{w \,|\, \exists u_1, .., u_k \in S, k \leqslant n \text{ such that } w = u_1.u_2...u_k\}$$

1. Show that for every S, $\mathrm{Eff}(S)$ is well defined. Is there an upper bound on $\mathrm{Eff}(S)$? Find a lower bound for $\mathrm{Eff}(S)$.

2. Let $k \geqslant 2$ be an integer. Out of every subset S satisfying $|S| = k$, maximise (and minimise) $\mathrm{Eff}(S)$.

Given a word v, we define $\ell_S(v)$ by:

$$\ell_S(v) = \min\{n \,|\, \exists u_1, .., u_n \in S, \text{ such that } v = u_1 u_2 ... u_k\}$$

3. Let S be a fixed subset. Describe a way to compute $\ell_S(v)$ for any $v \in S^*$.

Now we can define a notion of how similar two sets are. First, let us define the **distance** between two elements of $\{0,1\}^*$. For $u, v \in \{0,1\}^*$. The distance between the words u and v is $d(u,v) = 2^{-|P|}$, where P is the length of the longest word that appears at the beginning of both u and v, if $u \neq v$, and 0 if $u = v$. For example, let $u = 001$ and $v = 000$, then $P = 2$ and $d(u,v) = 1/4$. For two finite sets S_1 and S_2, we define:

$$d(S_1, S_2) = \max\left\{\max_{x\in S_1}\min_{y\in S_2} d(x,y)\,,\, \max_{x\in S_2}\min_{y\in S_1} d(x,y)\right\}$$

4. Let $(S_n)_{n\in\mathbb{N}}$ and $(T_n)_{n\in\mathbb{N}}$ be two sequences of subsets of $\{0, 1\}^*$ such that

$$\lim_{n\to\infty} d(S_n, T_n) = 0.$$

What can you say about the sequence $(\mathrm{Eff}(S_n) - \mathrm{Eff}(T_n))_n$?

We define the **density** of a set S as:

$$\mathcal{D}(S) = \lim_{n\to\infty} \frac{|S^{\leqslant n}|}{|S|^n}.$$

5. We wish to study this notion of density.

a) Is $\mathcal{D}(S)$ well-defined for every subset S? Give bounds as precise as possible for the value of $\mathcal{D}(S)$ for any subset S for which the density is well-defined.

b) Let $(S_n)_{n\in\mathbb{N}}$, $(T_n)_{n\in\mathbb{N}}$ be two sequences of subsets of $\{0, 1\}^*$ such that: $\lim_{n\to\infty} d(S_n, T_n) = 0$ and $\mathcal{D}(S_n) = \mathcal{D}(T_n) = 1$. Does $\mathrm{Eff}(S_n) - \mathrm{Eff}(T_n)$ have a limit?

c) Same question for $\mathcal{D}(S) = \ell$ for $\ell \in [0,1)$.

6. A subset is said to be **ℓ-sparse** if its density is a given $\ell \in [0,1]$. We denote by

$$\omega_{\ell,k} = \frac{|\{S \subset \{0,1\}^*, \ell\text{-sparse}, |S| \leqslant k\}|}{2^k}$$

and $\omega_\ell = \lim_{k\to\infty} \omega_{\ell,k}$.

a) Does ω_ℓ exist?
b) Give an example of 0-sparse subset.
c) For $k \geqslant 2$, compute ω_0.
d) Estimate as precisely as possible ω_ℓ for any $\ell \in (0,1]$.

7. Suggest and study other research directions.