ETEAM 2026

Problems for the third ETEAM

VERSION 1. UPDATED ON APRIL 10, 2026

Foreword

The problems that follow are difficult and are proposed by researchers and students in mathematics. To the best of the knowledge of the authors, they do not always admit a complete solution. However, they are accessible to high school students, i.e., the authors are certain that elementary research work can be carried out on these problems. The jury does not expect the candidates to solve a problem entirely, but rather to understand the issues, solve particular cases, identify difficulties, and suggest new directions of research. The questions are not always arranged in increasing order of difficulty. Finally, it is not necessary to solve all the problems: each team can reject a certain number of problems without penalty. Please refer to the rules for further details.

 If you have any question about the problems, please visit `https://eteam.tfjm.org/` or contact the Problem Committee via `baptiste.serraille@animath.fr`.

Contents

Keywords

1. Geometry 2. Graph theory 3. Optimisation, Analysis 4. Geometry, Combinatorics 5. Probability, Statistics, Programming 6. Dynamical systems, Analysis 7. Number theory, Optimisation, Algorithms 8. Geometry, Optimisation 9. Combinatorics

Notations

$\mathbb{R}, \mathbb{Z}$	The sets of real numbers and integer numbers, respectively
$[a,b)$	The set of all numbers $x \in \mathbb{R}$ such that $a \leqslant x < b$
$\mathbb{N}$	The set of strictly positive integer numbers $\{1,2,\dots\}$
$[\![n,m]\!]$	The set of integers $\{n,n+1,\dots,m\}$

1. Circle necklaces

Let n be a fixed integer with $n \geqslant 3$. A set of n circles $(C_1, C_2, ..., C_n)$ is called a **necklace** if the interiors of all the circles are pairwise disjoint, C_i is tangent from the outside to C_{i+1} at a point P_i for all $i = 1, \dots, n$ (where $C_{n+1} = C_1$), and all P_i lie on a base circle $\mathcal{C}$. We call the necklace **orthogonal** if all circles C_i are orthogonal to the base circle $\mathcal{C}$ (i.e. the tangents at P_i to $\mathcal{C}$ and to C_i are orthogonal). An example of an orthogonal necklace is given in Figure 1.

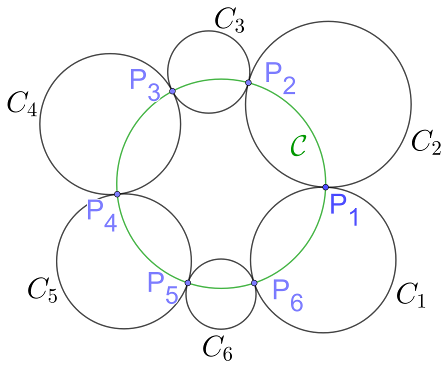

FIGURE 1. Example of an orthogonal necklace with $n = 6$.

1. Let $P_1, ..., P_n$ be n distinct points on a circle $\mathcal{C}$ in this cyclic order.

a) Does there exist a unique orthogonal necklace that has these points as tangent points? If so, give a construction.

b) Same question for a strictly non-orthogonal necklace.

An inversion in a circle with centre O and radius r maps a point P to the unique point P' on the half-line OP such that $|OP| \cdot |OP'| = r^2$. A transformation of the plane is called **inversive** if it is either an inversion in a circle or an axial reflection. We say that a necklace has a **symmetry of type** (i, j) (where $i, j \in \{1, \dots, n\}$ and $i \neq j$) if there is an inversive transformation fixing the base circle $\mathcal{C}$ and the circles C_i and C_j (each of them set-wise), and which is a symmetry of the necklace.

2. How many symmetries of type (i, j) for fixed i and j can there exist?

3. In this question, we consider n even.

a) Construct a necklace verifying that for all $i \in \{1, \dots, n\}$, there is a $j \in \{1, \dots, n\}$ such that the necklace has a symmetry of type (i, j).

b) Can you construct all necklaces of this form?

For two circles C_i and C_j, we denote by I_{ij} (resp. E_{ij}) the intersection point of the inner (resp. outer) common tangents.

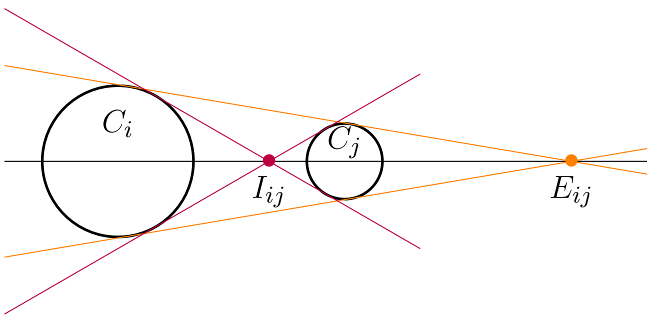

FIGURE 2. Intersection of common inner and outer tangents.

4. Suppose the necklace is orthogonal and has symmetry of type (i,j) (for some fixed $i<j$). If $I_{ij}\in C_{j-k}$ for some $k\in\{1,...,j-i\}$, is it true that $E_{ij}\in C_{j+k}$?

Now, we consider the discs D_i with boundary C_i (for $i=1,\dots,n$) and we add a finite number of new discs $D'_1,\dots,D'_m$ inside $\mathcal{C}$ such that the interiors of any two distinct discs among $\mathcal{D}=(D_1,\dots,D_n,D'_1,\dots,D'_m)$ are always disjoint. Such a configuration is called a **rigid filling** if all bounded connected components of $\mathbb{R}^2\setminus\bigcup_{D\in\mathcal{D}}D$ are triangles (with circular arcs as sides). An example is given in Figure 3. We call a rigid filling **minimal** if there is no rigid filling of the necklace with strictly less than m interior discs.

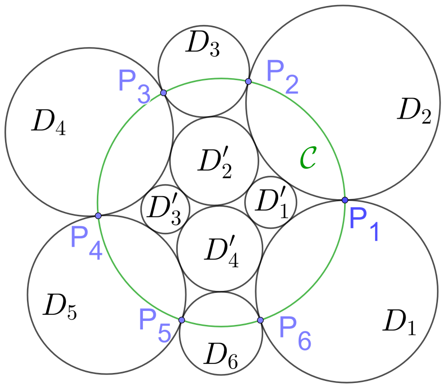

FIGURE 3. Example of an orthogonal necklace with rigid filling.

5. Find all pairs (n,m) such that there is a rigid minimal necklace with n boundary discs and m interior discs. Give a construction of these necklaces.

6. Characterise all necklaces that admit a rigid filling.

7. Suggest and study other research directions.

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2. LOST IN THE DUNGEON

A group of adventurers is exploring a dungeon made of rooms and corridors. Apart from the number of corridors that lead to them, the rooms are completely indistinguishable from one another. The corridors are also completely undistinguishable from one another. That is, when the adventurers come back to an already visited room, they have no other way than logical reasoning to figure out whether or not they went to a new room with the same number of corridors as the first one. However, in each room, the corridors are numbered clockwise from 1 to the number of corridors that lead to the room. In addition, the group has an absolute memory of all the rooms they visited, as well as the numbers of the corridors by which they entered and left.

We always assume that there is a finite number of rooms and corridors and that the dungeon is connected, that is, there is always a path going from any given room to any other one. A corridor can connect two (different) doors from the same room. Moreover, the dungeon does not need to be two-dimensional: a corridor may pass above or under another one without crossing it. (Otherwise stated, the graph representing the dungeon does not need to be planar.)

In addition, the group cannot split, all its members always have to stay in the same room or corridor at a given time. The adventurers enter the dungeon through a trapdoor which is completely undetectable once it is closed.

We call **move** the fact of going from a room to a neighbouring room through a corridor connecting them.

1. Do there exist two different dungeons that adventurers cannot distinguish in a finite number of moves?

The thief of the group was clever enough to block the entry trapdoor of the dungeon: from now on, the room from which the group starts can be distinguished from all the others because it is the only one where a trapdoor is visible.

2. The adventurers carry with them numbered coins that they can leave in the rooms to be able to distinguish them. Let $M(D,n)$ be the minimal number of moves needed to draw with certainty the map of the dungeon D with n numbered coins (and $M(D,n) = +\infty$ if it is impossible to draw with certainty the map of the dungeon D). The adventurers can move the coins. In the examples from Figure 4,

a) estimate $M(D,+\infty)$ as precisely as possible;
b) estimate $M(D,0)$ as accurately as possible.

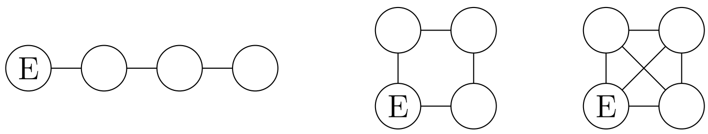

FIGURE 4. Three examples of dungeons. The letter E indicates the room from which one enters the dungeon. Note that the last dungeon is represented in a non planar way: The two diagonal corridors do not cross, one passes above the other.

3. For p, $q \in \mathbb{N}$, we denote by $M'(p,q,n)$ the maximal value (possibly infinite) that $M(D,n)$ can assume among all dungeons D with at most p rooms and q corridors. Estimate as precisely as possible, as a function of p and q, the following quantities (one can start with $p = q-1$ and $p = q$):

a) $M'(p,q,+\infty)$;
b) $M'(p,q,0)$.

4. The thief forgot to block the entry trapdoor: From now on, the entry room can no longer be distinguished from the other rooms. By chance, the adventurers remember that a sign outside the dungeon indicated the number of rooms in the dungeon. In the examples from Figure 4, estimate $M(D,k)$ for every $k \in \mathbb{N}$ (one can start with $k = n$ and $k = n-1$, where n is the number written on the sign).

5. In a new dungeon, the number of rooms is no longer indicated, but the adventurers know that a chest lies in one of the rooms. Is there a strategy allowing them to reach the chest for sure (not necessarily by exploring all the rooms), and if so, using how many moves at most?

6. Suggest necessary and/or sufficient conditions for two dungeons to be distinguishable by the adventurers. One will consider both the case where the entry room can be distinguished from the others and the case where it cannot.

7. Suggest and study other research directions.

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3. GARLANDS AND DRAPERIES

To make the opening ceremony of the ETEAM in Lausanne fancier, the LOC (Local Organisation Committee) wants to install a garland with math items on it. A garland with $n+1$ items is described by the $(n+1)$-tuple $\mathbf{x} = (x_0, \dots, x_n)$ with $x_i = \frac{i}{n}$, which encodes the horizontal positions of the items, and by the values $f(x_i)$ for some function $f\colon [0,1] \to \mathbb{R}$, describing the height at which the items hang. (All quantities and questions that follow depend on n. This

dependency is implicitly encoded in $\mathbf{x}$.) Due to constraints on the hooks on which the garland is hanging, the conditions $f(0)=0$ and $f(1)=1$ are imposed.
To find the fanciest possible arrangement of the garland, Deyan suggests to look for a configuration minimising the following quantity:

$$\mathbf{E}_{p,\mathbf{x}}(f) \coloneqq \sum_{i=1}^{n} |f(x_i)-f(x_{i-1})|^p.$$

1. Deyan's favorite number being 2, he first tries with $p=2$. What is the value of

$$\mathbf{E}_{2,\mathbf{x}} \coloneqq \inf\{\mathbf{E}_{2,\mathbf{x}}(f) \mid f\colon [0,1]\to\mathbb{R},\ f(0)=0,\ f(1)=1\}?$$

Is the infimum attained? If the infimum is achieved by some function $f\colon [0,1]\to\mathbb{R}$, is the tuple of values $(f(x_0),f(x_1),\dots,f(x_n))$ unique?

2. Deyan is not entirely satisfied, so he wants to try with other values of p. Answer **1.** for other values of $p\in(0,+\infty)$.

As the result is still not satisfactory, Ismail suggests to add a term in front of the height of the items. More precisely, given a function $V\colon [0,1]\to\mathbb{R}$, he suggests to rather consider the quantity

$$\mathbf{E}_{p,\mathbf{x},V}(f) \coloneqq \sum_{i=1}^{n} |f(x_i)-f(x_{i-1})|^p + f(x_i)V(x_i).$$

3. Answer **1.** and **2.** for $\mathbf{E}_{p,\mathbf{x},V} \coloneqq \inf\left\{\mathbf{E}_{p,\mathbf{x},V}(f) \mid f\colon [0,1]\to\mathbb{R},\ f(0)=0,\ f(1)=1\right\}$.

Théo, not being a fan of garlands because he finds one-dimensional objects boring, suggests to rather use draperies made of tiny triangular napkins. Let $S(a,b,c)$ denote the area of a triangle with sidelengths a, b, and c (one can use for example Héron's formula). For a function $f\colon [0,1]^2\to\mathbb{R}$, we define

$$\mathbf{A}_n(f) \coloneqq \sum_{i,j=0}^{n-1} S\left(\ell_{i,j,i,j+1},\ell_{i,j,i+1,j},\ell_{i+1,j,i,j+1}\right) + \sum_{i,j=1}^{n} S\left(\ell_{i,j,i,j-1},\ell_{i,j,i-1,j},\ell_{i-1,j,i,j-1}\right),$$

where we denote $f_{i,j} \coloneqq f(x_i,x_j)$ and

$$\ell_{i,j,k,l} \coloneqq \sqrt{\frac{1}{n^2}+(f_{i,j}-f_{k,l})^2}$$

for the sake of conciseness of notation. This represents the area of the drapery whose triangular pieces are hanged at height $f(x_i,x_j)$ above the point (x_i,x_j), and that Théo suggests to minimise to obtain nice looking motives.

4. Once again, the LOC needs to take into account the availability of hooks on the walls. Let

$$B\subset(\{0\}\times\{0,1,\dots,n\})\cup(\{n\}\times\{0,1,\dots,n\})\cup(\{0,1,\dots,n\}\times\{0\})\cup(\{0,1,\dots,n\}\times\{n\})$$

be a subset of the set of indices that encode the boundary of the drapery. Finally, we let $\mathbf{g}_B=(g_{i,j})_{(i,j)\in B}$ be a set of prescribed heights corresponding to the indices $(i,j)\in B$. Is the infimum

$$A_{n,\mathbf{g}_B} = \inf\{\mathbf{A}_n(f) \mid f\colon [0,1]^2\to\mathbb{R},\ f_{i,j}=g_{i,j} \text{ for every } (i,j)\in B\}$$

attained?

5. In the context of the previous question, compute or estimate as precisely as possible the value $A_{n,\mathbf{g}_B}$, as well as the minimiser(s) when the infimum is attained. The following particular cases are of special interest:

a) $B=\varnothing$;
b) $B=\{(0,0)\}$;
c) $B=\{(0,0),(n,n)\}$;
d) $B=\{(0,0),(0,n)\}$;

e) $B = \{0\} \times \{0, 1, \ldots, n\}$.

The limiting behaviour when $n \to +\infty$ is of special interest, either in the general case or in the above special situations.

6. Suggest and study other research directions.

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4. Rubber band artwork

Eve is an artist who creates art installations from hammering nails into wooden boards and putting rubber bands around them. Being a completionist, she wants to add rubber bands around every subset of nails as long as the rubber band remains convex. The rubber bands she has at home are labelled with an integer. She starts by placing a rubber band labelled 1 around each single nail. For every other rubber band, she imposes the following rule:

> The sum of the labels of each rubber band and all the rubber bands it contains is 1.

Here, a rubber band B_1 contains a rubber band B_2, if all the nails that B_2 encloses are also enclosed by B_1.

Example: She has put three nails in the positions $a = (0,0)$, $b = (1,0)$ and $c = (0,1)$. If the rubber band around $\{a,b,c\}$ is labelled with 1 and the rubber bands around $\{a,b\}$, $\{a,c\}$, and $\{b,c\}$ have the label -1 then the rule is satisfied. If she had instead chosen the position $c = (2,0)$ and if the rubber band around $\{a,b\}$ and $\{b,c\}$ still have the label -1, while the rubber band around $\{a,b,c\}$ would be labelled 0, then the rule would again be satisfied. There is no rubber band that only encloses $\{a,c\}$ alone, since it would also enclose the nail at b.

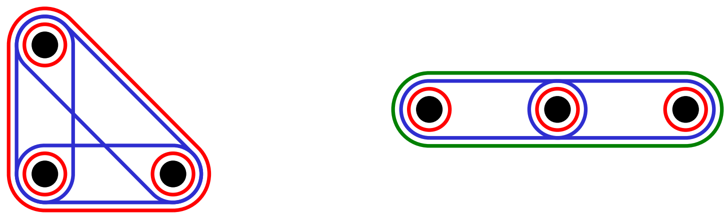

Figure 5. Rubber bands for the two examples above. Bands with label 1 are red, bands with label 0 are green and bands with label -1 are blue.

1. Eve starts by building two prototypes. Compute a choice of labels for all bands when the nails are put into the following positions:

a) n nails in a row: $L(n) = \{(i,0) : i \in \{1, \ldots, n\}\}$;

b) at the vertices of a regular n-gon: $R(n) = \{(\cos(\frac{2\pi i}{n}),\ \sin(\frac{2\pi i}{n})) : i \in \{0, \ldots, n-1\}\}$.

2. For which finite subsets $X \subseteq \mathbb{R}^2$ can Eve choose a labelling of the rubber bands? When is it unique?

Being satisfied with her prototypes, Eve decides to start a series of art installations. For each $n \in \mathbb{N}$, she put the nails at

$$C(n) := \{(i,0) \in \mathbb{Z}^2 : 0 \leqslant i \leqslant n\} \cup \{(0,i) \in \mathbb{Z}^2 : 1 \leqslant i \leqslant n\}.$$

3. Compute or give an estimate for which $n \in \mathbb{N}$ her art project becomes infeasible to do for nails at positions $C(n)$, given that the world production of rubber in 2025 was roughly $1.49 \cdot 10^{10}$kg and the mass of a rubber band is about 1g.

From now on, Eve adds more nails and considers them at the following positions:

$$T(n) := \{(i,j) \in \mathbb{Z}^2 : 0 \leqslant i \leqslant j \leqslant n\}.$$

4. Reconsider the previous question for $T(n)$.

5. To save some rubber bands, she decides to only add rubber bands with a non-zero label. Describe as precisely as possible the shapes of rubber bands with non-zero labels as well as the subset $A \subseteq \mathbb{Z}$ of labels that occur.

6. For $i \in \mathbb{N}$ let a_i be the number of rubber bands with non-zero label that enclose i nails. Given n, estimate a_i as precisely as possible. How large can she make her art installation before she hits the world's rubber production limit when only considering non-zero labels?

7. Consider **6.** when instead she put nails at

$$Q(n) := \{(i,j) \in \mathbb{Z}^2 : 0 \leqslant i \leqslant n, 0 \leqslant j \leqslant n\}.$$

8. She throws a dart at the board. Depending on where it ends up, what is the sum of the labels of the rubber bands that enclose the dart? Answer this question for $L(n)$, $R(n)$, $T(n)$, $Q(n)$, or in general for any art installation consisting of finitely many nails (at integer coordinates or even unrestricted).

9. Suggest and study other research directions.

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5. EXPLORING RANDOMNESS

Nalini wants to create programmes on her computer that generate random strings of length $n \geqslant 1$ from a given alphabet of symbols $A = \{s_1, s_2, \ldots, s_N\}$, where $N \geqslant 2$. To evaluate their efficiency, she needs to construct a function that assigns to each string a **randomness score** between 0 and 1 that measures how close the string is to being uniformly distributed. This function should take into account not only the frequency of each symbol, but also the order in which the symbols appear in the string.

For any integers k and n with $1 \leqslant k \leqslant n$, we denote by $p(n,k)$ the probability of finding at least k consecutive identical symbols in a string of length n, picked uniformly at random.

1. Find $p(n,2)$ for any given integer $n \geqslant 2$.

2. For $N = 2$, compute $p(200,6)$.

3. Consider a sequence of positive integers $(k_n)_{n\geqslant 1}$ for which there is some $\varepsilon > 0$ satisfying $1 \geqslant \frac{k_n}{n} > \varepsilon$ for all n. Analyse the behaviour of the sequence $(p(n,k_n))_{n\geqslant 1}$ as n goes to $+\infty$. Try to generalise this for other growth rates of $(k_n)_{n\geqslant 1}$ (e.g. $(k_n)_{n\geqslant 1}$ is asymptotically equivalent to $\sqrt{n}$).

4. Find a general formula for $p(n,k)$, or give estimates. What is the limit of $p(n,k)$ as n goes to infinity and k stays constant?

5. Given $n \geqslant 1$ and $\alpha \in [0,1]$, find lower and upper bounds for the maximum value of k with $p(n,k) \geqslant \alpha$.

Any string S of n symbols can be viewed as a concatenation of blocks with identical symbols. Denote by $\mathrm{B}_n(S)$ the number of blocks of a string S of length n. For example, if $A = \{0,1\}$ and $S = 1110011101010000$ is a string of 16 symbols, we have $S = 111|00|111|0|1|0|1|0000$ and therefore $\mathrm{B}_{16}(S) = 8$ blocks with identical symbols.

6. Compute the expectation of B_n, denoted by $\mathbb{E}[\mathrm{B}_n]$.

For any string S of length n, denote by $\mathrm{L}_n(S)$ the length of the largest block with consecutive identical symbols from S.

7. Compute or estimate the expectation of L_n and analyse as precisely as you can the growth rate of the sequence $(\mathbb{E}[\mathrm{L}_n])_{n\geqslant 1}$ as n goes to $+\infty$.

Nalini defines the **randomness score** of a string S of length n with symbols from A to be

$$\mathrm{R}_n(S) = \left[1 - \frac{N}{N-1}\sum_{i=1}^{N}\left(p_i - \frac{1}{N}\right)^2\right]\cdot\exp\left(-\frac{|\mathrm{L}_n(S)-\mathbb{E}[\mathrm{L}_n]|}{\mathbb{E}[\mathrm{L}_n]}\right)\cdot\exp\left(-\frac{|\mathrm{B}_n(S)-\mathbb{E}[\mathrm{B}_n]|}{\mathbb{E}[\mathrm{B}_n]}\right)\in[0,1],$$

where p_i is the relative frequency of the symbol s_i in the string S. This randomness score takes into account symbol frequency, the largest block with identical symbols, and the number of blocks.

8. Devise an algorithm to compute $\mathrm{R}_n(S)$ for a given string $S \in A^n$. Is it possible to determine what proportion of strings from A^n have a randomness score of at least 95%, for a large n (e.g. $n = 200$)?

9. Identify limitations of the given randomness score function in detecting random strings, and propose improved definitions of it. Suggest and study other research directions.

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6. SQUIRRELS PLAYING IN A TREE

We are interested in a jolly company of squirrels living in a huge tree. The tree is N meters high and, at each integer height, each branch splits into two branches, see Figure 6. There are n squirrels, where $n \leqslant 2^N$, and each squirrel has some amount of energy $\alpha_i \in \mathbb{R}_+$ $(i = 1,\ldots,n)$. The favourite game of the squirrels goes as follows. At the beginning, each squirrel is alone on a leaf of the tree. Some leaves may remain empty. Each squirrel goes down the tree at the same time, and, at each integer height, they may meet another squirrel. On that occasion, they play *bullet chess* and the winner goes on going down the tree with some amount of energy $F(\beta,\gamma)$, where β is the former energy of the winning squirrel and γ is the former energy of the losing squirrel. If a squirrel meets nobody on the other branch, it goes on without changing its amount of energy.

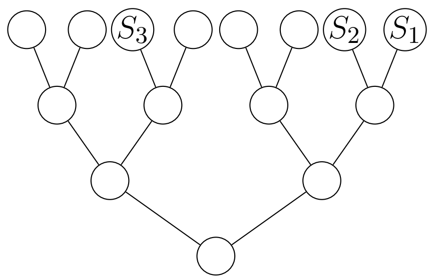

FIGURE 6. Tree with height $N = 3$. At time 1, S_1 and S_2 meet, and at time 3, the winner meets S_3.

1. Assume that the squirrel with more energy always wins and keeps its energy, i.e., $F(\beta,\gamma) = \beta$, with $\beta \geqslant \gamma$. If $\beta = \gamma$, the winner is chosen at random with equal probability. In terms of $\alpha_1,\ldots,\alpha_n$, what is the amount of energy of the squirrel at the bottom of the tree?

2. Assume that the squirrel with more energy always wins, but after the game the squirrels share their energy: $F(\beta,\gamma) = \frac{\beta+\gamma}{2}$. In terms of $\alpha_1,\ldots,\alpha_n$, which values can take the amount of energy of the squirrel at the bottom of the tree? One can start by dealing with the situation where the squirrels are initially on the n leftmost leaves.

From now on, we assume that initially all squirrels carry the same amount of energy $\alpha_{\text{init}} \in [0,1]$. We also assume that the squirrel with more energy is more likely to find a good strategy, and thus to win. Namely, if two squirrels with energy β_1 and β_2 meet, squirrel i wins with probability $\frac{\beta_i}{2}$ $(i=1,2)$. There is a probability $1-\frac{\beta_1+\beta_2}{2}$ of a draw, in which case both squirrels stop playing and descending.

3. Assume that the winner (if there is one) keeps all its energy, so $F(\beta,\gamma)=\beta$. What is the probability that the root is reached by a squirrel in the following cases?

a) When all leaves are occupied by a squirrel.
b) When the $0 \leqslant n \leqslant 2^N$ leftmost leaves are occupied, and the others are empty.

In both the above cases, what happens in the limit $N \to +\infty$?

Bullet chess is tiring. The rule when two squirrels meet is now the following. Fix $p \in (0,1)$ and $\varepsilon \in (0,1)$: With probability p, the winner (if there is one) loses a proportion ε of its energy after the game, i.e., $F(\beta,\gamma)=(1-\varepsilon)\beta$. With probability $1-p$, the winner keeps its energy, so $F(\beta,\gamma)=\beta$. In addition, if a squirrel does not meet anyone, it can choose to do nothing or duplicate, with probability $1/2$. In the former case, it keeps its energy and goes on, and in the latter case it plays bullet chess against its double, and the same rule as for two squirrels playing applies (with $\beta=\gamma$).

4. Assume that the $\lfloor 2^N x \rfloor$ leftmost leaves are occupied by squirrels at the initial time (for some proportion $x \in [0,1]$). Let $u_N(x)$ be the probability that a squirrel reaches the bottom of the tree. Does $u_N(x)$ increase (or non-decrease) with x?

5. Estimate as precisely as possible the limit of $u_N(x)$ as $N \to +\infty$.

6. Now, when a squirrel meets nobody at a new branch, then, if it chooses to do nothing (which occurs with probability $1/2$), it rests and its energy goes from α to $(1+\delta)\alpha$, for some fixed $\delta > 0$. (The same rule as in question 4 applies if it plays against its double, which occurs with probability $1/2$.) In terms of δ, how does the answer to the previous question change?

7. Suggest and study other research directions.

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7. Factor and conquer

The robot $\mathbb{R}^2$-D_2 has a powerful computational tool which allows it to perform multiplications, but some parts of its computation have a cost.
Throughout the problem, all numbers are positive integers written in base 10. Let $k \geqslant 2$ and $a_1,\ldots,a_k$ be integers strictly greater than 1. To compute the product $a_1 \times \cdots \times a_k$, the robot may perform the following operations:

- It may multiply two currently available integers a and b using the usual long multiplication algorithm (also called the grade-school multiplication algorithm). This operation has a cost described below.
- It may factor any currently available integer a as $a=uv$ where u and v are positive integers. The integer a is then replaced by the two newly available integers u and v. This operation is free.
- It may reorder the currently available factors in any way. This operation is also free.

The goal is to compute the product $a_1 \times \cdots \times a_k$ with the smallest possible total cost denoted by $f(a_1,\ldots,a_k)$.

Cost of a direct multiplication. Let a and b be two positive integers, write $b=\sum_{j=1}^{n} b_j 10^{j-1}$ in base 10. To compute the product $a \times b$, the robot uses the long multiplication algorithm (see also the Wikipedia page Multiplication algorithm):

a) For each digit b_j of b, it forms the partial product $a \times b_j$ and shifts it to the left by $j-1$ positions. This operation is free.

b) It then adds all the shifted partial products, this operation has a cost. More precisely, the addition is performed column by column from right to left. In a given column, terms equal to 0 do not contribute to the cost. If exactly r nonzero terms must be added in that column (with $r \geqslant 1$), then the cost of that column is $r-1$. A possible carry from the previous column counts as one additional nonzero term.

The total cost of this long multiplication is denoted by $g(a,b)$.

Example. To compute 123×236, the long multiplication is

$$\begin{array}{ccccc} & & 1 & 2 & 3 \\ \times & & 2 & 3 & 6 \\ \hline & & 7 & 3 & 8 \\ & 3 & 6 & 9 & \cdot \\ 2 & 4 & 6 & \cdot & \cdot \\ \hline 2 & 9 & 0 & 2 & 8 \end{array}$$

The cost is computed as follows:

- Units column: $8+0+0$ has a cost of 0, with no carry.
- Tens column: $3+9+0$ has a cost of 1, with carry 1.
- Hundreds column: $7+6+6+1_{\text{carry}}$ has cost 3, with carry 2.
- Thousands column: $0+3+4+2_{\text{carry}}$ has cost 2, with no carry.
- Ten-thousands column: $0+0+2$ has cost 0.

Hence, $g(123,236) = 0+1+3+2+0 = 6$. But a better way to compute this product is as follows:

a) Factor $123 = 3 \times 41$ and $236 = 4 \times 59$ (cost 0).
b) Multiply $3 \times 4 = 12$ (cost 0).
c) Multiply $12 \times 59 = 108 + 600 = 708$ (cost 1).
d) Multiply $708 \times 41 = 708 + 28320 = 29028$ (cost 2).

This strategy has total cost 3. Therefore, $f(123,236) \leqslant 3$.

1. Find all positive integers n such that

a) $g(7,n) = g(n,7)$;
b) $g(6,n) = g(n,6)$.

2. For a given positive integer N, estimate the number

a) $S(N)$ of pairs (a,b) with $2 \leqslant a,b \leqslant N$ such that $g(a,b) = g(b,a)$;
b) $\Delta(N)$ defined as the maximum of all $\left|g(a,b) - g(b,a)\right|$ for $2 \leqslant a,b \leqslant N$.

The factorisation module of the internal programme of $\mathbb{R}^2$-D_2 is undergoing maintenance. Now, it has only access to a degraded algorithm. To compute the product $a_1 \times \cdots \times a_k$, the robot can only reorder the factors or multiply two currently available integers a and b using the long multiplication algorithm with cost $g(a,b)$. Denote by $h(a_1,\ldots,a_k)$ the cost of the most efficient way to perform the product $a_1 \times \cdots \times a_k$ under this degraded algorithm.

3. In this question, $\mathbb{R}^2$-D_2 can only use the degraded algorithm.

a) In how many ways can it perform the product $a_1 \times \cdots \times a_k$?
b) For which integers k does there exist a choice of positive integers $a_1,\ldots,a_k$, each having at most k digits, such that all direct multiplication schemes have pairwise distinct total costs?

The maintenance is now finished, and the factorisation module is fully available again: $\mathbb{R}^2$-D_2 can now use the full power of its algorithm.

4. For a fixed $k \geqslant 2$, determine the smallest possible value of $f(a_1,\ldots,a_k)$ where $a_1,\ldots,a_k$ are pairwise distinct and

a) all the a_i's are greater than some given constant $A > 1$;
b) all the a_i's are prime.

5. Let $m \geqslant 0$ be an integer.

a) Does there exist a pair (a,b) of integers strictly greater than 1 such that $f(a,b) = m$?
b) Among all the pairs (a,b) such that $f(a,b) = m$, how large can $g(a,b)$ be?

6. For a fixed $k \geqslant 2$, find a class of k-tuples $(a_1,\ldots,a_k)$ for which

a) $f(a_1,\ldots,a_k) < h(a_1,\ldots,a_k)$;
b) alternating factoring and products is genuinely useful, i.e., $f(a_1,\ldots,a_k)$ is strictly smaller than the cost of the most efficient algorithm in which one may factor only the initial integers $a_1,\ldots,a_k$, and must then use the degraded algorithm.

7. Suggest and study other research directions.

* * *

8. THE PEN ULTIMATE PROBLEM

Bob the Builder wants to enclose a pen for his rabbits. He found a cheap set of fences, but their lengths are a bit odd. In the set there are $N \geqslant 3$ fences with strictly positive lengths $0 < \ell_1 \leqslant \ell_2 \leqslant \ldots \leqslant \ell_N$. Bob wants to construct a figure with them that has maximal area. He can put the fences however he wants, not necessarily end-to-end or non-crossing. If the constructed figure encloses multiple regions with disjoint interiors, then we add their areas.

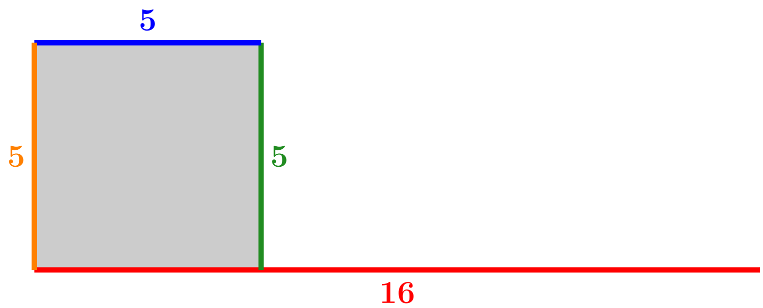

FIGURE 7. Pen built with four fences of lengths $0 < 5 \leqslant 5 \leqslant 5 \leqslant 16$.

1. Let $N = 3$.

a) What is the maximum area of a pen when the fences have lengths $\ell_1 = 1, \ell_2 = 2$ and $\ell_3 = 3$?
b) Treat the general case with three fences.

2. In this question, N is general and Bob is allowed to cut pieces from the fences and reassign them to other fences, provided that each fence remains a single straight segment (i.e., no bends or corners are introduced). In particular, the total number of fences N and the perimeter $\mathcal{P} = \ell_1 + \ell_2 + \cdots + \ell_N$ remain unchanged. What is the maximal area of a pen that he can construct?

3. Let $N = 4$.

a) Bob tries to construct a quadrilateral pen using his fences without any loss of material. What is the maximum possible area in this case?
b) Compute or give estimates for the maximum area for general fence lengths in the case $N = 4$ (loss of material is allowed).

4. Reconsider the previous question for $N \geqslant 5$.

We now introduce a profit functional J. For any pen $\mathcal{F}$ that Bob can make with the given N fences we set

$$J(\mathcal{F}) = a \cdot \mathrm{Area}(\mathcal{F}) - \ell \cdot \mathrm{Loss}(\mathcal{F}).$$

Here, $a > 0$ represents the profit per unit of area and $\ell > 0$ is the cost of the fence per unit of length. By Area($\mathcal{F}$) we mean the area enclosed by the fences of $\mathcal{F}$, and by Loss($\mathcal{F}$) we understand the total length of the (parts of) fences of $\mathcal{F}$ that are not included in the boundary of the enclosed area.

5. Try to maximise the profit functional J. One could start with $N = 3$ and $N = 4$.

6. Bob can cut the fences as in **2.**, but now each cut comes with a cost $c > 0$. Maximise the functional

$$I(\mathcal{F}) = a \cdot \mathrm{Area}(\mathcal{F}) - \ell \cdot \mathrm{Loss}(\mathcal{F}) - c \cdot \mathrm{Cuts}(\mathcal{F}),$$

where by Cuts($\mathcal{F}$) we mean the number of cuts made by Bob to obtain the pen $\mathcal{F}$ from the given N fences.

7. Suggest and study other research directions.

* * *

9. MAGIC CROSSES

Lea and Leo explore configurations of crosses on a square grid, subject to some conditions. They consider a rectangle of size $m \times n$ (with $m, n \in \mathbb{N}_{\geqslant 2}$), decomposed into small squares of size 1×1. They mark certain squares with a cross, which they call a cross configuration. They also consider the shape L which consists of 3 squares: a middle square to which you add a square to the top and to the right (see left of Figure 8). The shape L, as well as any other shape considered later, is not allowed to rotate and must be placed to match the squares of the grid. A cross configuration is called **magical** if the following condition is satisfied:

(1) Wherever you put the shape L completely inside the rectangle, it covers an *even* number of crosses.

Figure 8 shows an example of a magic cross configuration.

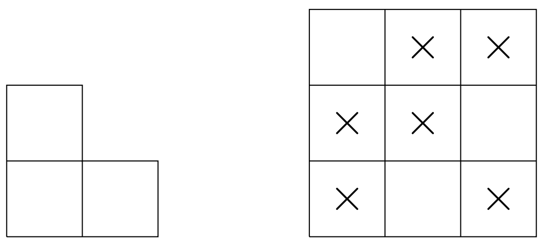

FIGURE 8. The shape L (left) and an example of a magic cross configuration for $m = n = 3$ (right). Note that there are 4 ways to put the shape L inside the 3×3 grid without turning it.

1. For which couples (m, n) are there at least two magical cross configurations?

2. Compute or estimate the number of magical cross configurations for fixed m and n.

3. What is the minimal number of crosses in a non-empty magical cross configuration?

In what follows, two crosses are said to be **neighbours** when the squares they belong to share an edge. Two crosses are **connected** if there is a chain of neighbouring crosses that connects them. A maximal set of crosses where any two are connected is called a **connected component**.

4. How many connected components can a magical cross configuration have?

Lea proposes to modify the rules slightly: instead of playing on the rectangle, she suggests that they play on an infinite, but periodic grid. This means that a cross configuration has to be invariant under the shift by the vector $(0, m)$ and by the vector $(n, 0)$. Condition (1) then applies to any position of the shape L on the infinite grid.

5. Reconsider **2.**, **3.**, and **4.** in this infinite periodic setting.

Leo proposes to change condition (1) slightly:

(1') Wherever you put the shape L on the plane, it covers *exactly one* cross.

6. Reconsider **2.**, **3.**, and **4.** (on the infinite periodic grid) under condition (1').

Lea proposes to change the shape L into a "plus" shape, denoted P, consisting of five squares, one central square and all of its four neighbours. Condition (1) is then replaced by

(2) Wherever you put the shape P on the plane, it covers an *odd* number of crosses.

7. Reconsider **2.**, **3.**, and **4.** under condition (2).

8. Suggest and study other research directions.